

On partial asymptotic stability and instability. II (The method of limiting equation)

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1. Introduction

In [1] we established criteria on the partial asymptotic stability and instability based on Ljapunov functions with semidefinite derivatives not requiring boundedness of solutions. We proved an alternative for every solution of an autonomous system saying that either all the controlled coordinates tend to zero or the vector of the uncontrolled coordinates tends to infinity as $t \rightarrow \infty$ (see [1], Theorem 3.1). Combining this result with additional hypotheses on the *Ljapunov function* we found sufficient conditions for the partial asymptotic stability and instability of the zero solution. By the aid of these theorems we could study stability properties of equilibrium positions of certain mechanical systems in the presence of dissipative forces. However, as it was mentioned in [1], to apply the alternative to certain mechanical systems one needs additional conditions of other types. For example, consider a material point moving on a surface in a constant field of gravity in the inertial frame of reference $Oxyz$ (Oz directed vertically upward) and subject to viscous friction [1]. Let the point be constrained to move on the surface of the equation $z = (1/2)y^2 \times [1 + 1/(1+x^2)]$. Theorems in [1] cannot be applied to prove asymptotic y -stability for the equilibrium position $x=y=0$. Nevertheless, it is reasonable to conjecture that the equilibrium position possesses this property. For, if a motion $(x(t), y(t))$ is bounded, then $|y(t)| \rightarrow 0$ as $t \rightarrow \infty$ (see [1], Theorem A). On the other hand, if $|x(t)| \rightarrow \infty$ as $t \rightarrow \infty$, then the motion $(x(t), y(t))$ is "asymptotically near" to a motion of the point on the surface of the equation $z = (1/2)y^2$, for which the equilibrium position $x=y=0$ is asymptotically y -stable.

The purpose of this paper is to establish partial asymptotic stability and instability of the zero solution of such system whose *right-hand side* allows a limiting process as the vector of the uncontrolled coordinates tends to infinity in norm.

The paper is organized as follows. In Section 3 we treat such autonomous

system whose right-hand side has a uniform limit as the vector of uncontrolled coordinates tends to infinity in norm. In Section 4 results of the previous section will be applied to study partial stability properties of the equilibrium position with respect to all generalized velocities and some of generalized coordinates in the scleronomous holonomic mechanical systems being under the action of viscous friction. The method to be presented works also for the nonautonomous differential systems. Section 5 is devoted to this generalization. Whilst Section 3 is based upon the standard sphere of concepts of stability theory and is selfcontained, Section 5 is strongly connected with a recent topic of the theory of limiting equations developed by Z. ARTSTEIN [2]—[4], some of whose results are necessary preliminaires for applying our main theorem.

2. A nonautonomous invariance principle

All the necessary notations and definitions have been introduced in [1] (see Section 2) excepting the following one. Consider the system of differential equation

$$(2.1) \quad \dot{x} = X(x, t) \quad (t \in R_+, x \in R^k),$$

where the function X is continuous in x , is measurable in t , and satisfies the Carathéodory condition locally on the set Γ_y . Let us given a Ljapunov function $V: \Gamma'_y \rightarrow R$ (for $\Gamma'_y \subset \Gamma_y \subset R^m \times R^n \times R_+$ see [1], Sec. 2). For $c \in R$ denote by $V_m^{-1}[c, \infty)_0$ the set of the points $y \in R^m$ for which there exists a sequence $\{(y_i, z_i, t_i)\}$ such that $y_i \rightarrow y$, $|z_i| \rightarrow \infty$, $t_i \rightarrow \infty$, $V(y_i, z_i, t_i) \rightarrow c$ and $\dot{V}(y_i, z_i, t_i) \rightarrow 0$ as $i \rightarrow \infty$. Obviously, $V_m^{-1}[c, \infty)_0$ is closed relative to Γ'_y .

We shall need the following nonautonomous invariance principle even in Section 3 where the basic differential system is assumed to be autonomous.

Theorem A. [5]—[7] *Assume that for every compact set $K \subset R^k$ there is a $\mu_K \in \mathcal{K}$ such that if $u: [\alpha, \beta] \rightarrow K$ is continuous then*

$$(2.2) \quad \left| \int_{\alpha}^{\beta} X(u(t), t) dt \right| \leq \mu_K(\beta - \alpha).$$

If $V: \Gamma'_x \rightarrow R$ is a Ljapunov function bounded below, and $\varphi: [t_0, \infty) \rightarrow R^k$ is a solution of (2.1) for which $|\varphi(t)| \leq H'' < H'$ holds for all $t \geq t_0$, then $\Omega_x(\varphi)$ is contained in a component of $V_k^{-1}[c, \infty)_0$ for some constant c .

In order to make Section 3 selfcontained we sketch the proof. Since V is locally Lipschitzian, the function $v(t) = V(\varphi(t), t)$ is locally absolutely continuous and

$$(2.3) \quad \frac{d}{dt} v(t) = \dot{V}(\varphi(t), t) \leq 0$$

for almost all $t \geq t_0$. Thus $v(t)$ is nonincreasing and $v(t) \rightarrow c$ as $t \rightarrow \infty$ for some constant c . Suppose that the statement is false. Then there exist $p \in \Omega_x(\varphi)$ and $\varepsilon > 0$ such that $\bar{B}_k(p, 2\varepsilon) \cap V_k^{-1}[c, \infty]_0 = \emptyset$, where $\bar{B}_k(p, 2\varepsilon)$ denotes the closed ball in R^k with center p and radius 2ε . Obviously,

$$(2.4) \quad \limsup_{T \rightarrow \infty} \{\dot{V}(\varphi(t), t) : t \geq T, \varphi(t) \in \bar{B}_k(p, 2\varepsilon)\} < 0,$$

thus, however large the time T^* may be, the point $\varphi(t)$ cannot be contained in the set $\bar{B}_k(p, 2\varepsilon)$ for all $t \geq T^*$ since v is bounded below. Therefore, $\varphi(t)$ enters $\bar{B}_k(p, \varepsilon)$ and leaves $B_k(p, 2\varepsilon)$ infinite number of times. In view of (2.2)–(2.4) this means that v is not of bounded variation, which is a contradiction.

3. Autonomous equations

Consider the differential system

$$(3.1) \quad \dot{x} = X(x) \quad (x \in R^k; X(0) = 0),$$

where $X : G_y \rightarrow R^k$ is continuous. By the partition $x = (y, z)$ ($y \in R^m, z \in R^n; 1 \leq m \leq k, n = k - m$) the system (3.1) can be written in the form

$$(3.2) \quad \dot{y} = Y(y, z), \quad \dot{z} = Z(y, z).$$

Throughout this section we assume that $Y(y, z) \rightarrow Y_*(y)$ uniformly in $y \in \bar{B}_m(H')$ as $|z| \rightarrow \infty$.

Theorem 3.1. *Suppose that there is a Ljapunov function $V : G'_y \rightarrow R$ of (3.2) satisfying the following conditions:*

- (i) V is positive y -definite;
- (ii) for every $c > 0$ the set $(\dot{V}_{(3.2)})^{-1}(0) \cap V^{-1}(c)$ contains no complete trajectory of (3.2), and
- (iii) the set $V_m^{-1}[c, \infty]_0$ contains no complete trajectory of the system

$$(3.3) \quad \dot{y} = Y_*(y)$$

except the origin of R^m .

Then the zero solution of (3.2) is asymptotically y -stable.

Proof. Since V is positive y -definite and $\dot{V}_{(3.2)}(y, z) \leq 0$ on G'_y , the zero solution of (3.2) is y -stable (see [8], p. 15), i.e. for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|x_0| < \delta(\varepsilon)$ implies $|y(t; x_0)| < \varepsilon$ for all $t \geq 0$. Let $0 < \varepsilon_0 < H'$ and define $\sigma = \delta(\varepsilon_0) > 0$. We shall prove that for every $x_0 \in B_k(\sigma)$ we have $|y(t; x_0)| \rightarrow 0$ as $t \rightarrow \infty$.

Let $x = \varphi(t) = (\psi(t), \chi(t))$ be a solution of (3.2) such that $\varphi(0) \in B_k(\sigma)$. The function $v(t) = V(\varphi(t))$ is nonincreasing and nonnegative, hence $v(t) \rightarrow v_0 \geq 0$ as $t \rightarrow \infty$. If $v_0 = 0$ then $|\psi(t)| \rightarrow 0$ as $t \rightarrow \infty$ since V is positive y -definite. Assume that $v_0 > 0$. By Theorem 3.1 in [1], this assumption together with (ii) imply

$$(3.4) \quad \lim_{t \rightarrow \infty} |\chi(t)| = \infty.$$

Consider the system

$$(3.5) \quad \dot{y} = Y(y, \chi(t)) \quad (y \in B_m(H'), t \in R_+)$$

and the function $U: B_m(H') \times R_+ \rightarrow R$ defined by $U(y, t) = V(y, \chi(t))$. Obviously,

$$(3.6) \quad \dot{U}_{(3.5)}(y, t) = \dot{V}_{(3.2)}(y, \chi(t)) \leq 0,$$

therefore U is a Ljapunov function of (3.5) and $u(t) = U(\psi(t), t) \rightarrow v_0$ as $t \rightarrow \infty$. The function $y = \psi(t)$ is a solution of equation (3.5), whose right-hand side is bounded for $(y, t) \in \bar{B}_m(H') \times R_+$, and $|\psi(t)| \leq \varepsilon_0 < H'$ for all $t \geq 0$. By Theorem A' in Section 2 we have the inclusion $\Omega_y(\psi) \subset U_m^{-1}[v_0, \infty]_0$. Furthermore, in view of (3.4) and (3.6), $U_m^{-1}[v_0, \infty]_0 \subset V_m^{-1}[v_0, \infty]_0$. Taking into account the obvious fact that the positive y -limit set $\Omega_y(\varphi)$ of the solution $x = \varphi(t)$ of (3.2) coincides with the positive limit set $\Omega_y(\psi)$ of ψ , being a solution of (3.5), we obtain

$$(3.7) \quad \Omega_y(\varphi) = \Omega_y(\psi) \subset V_m^{-1}[v_0, \infty]_0.$$

On the other hand, property (3.4) implies that $Y(y, \chi(t)) \rightarrow Y_*(y)$ uniformly in $y \in \bar{B}_m(H')$ as $t \rightarrow \infty$. Thus (3.3) is the limit equation of (3.5) and $\Omega_y(\psi)$ is semiinvariant with respect to (3.3) (see [8], p. 304). Now we can conclude the proof by showing that $\Omega_y(\varphi) = \{0\}$, i.e. $|\psi(t)| \rightarrow 0$ as $t \rightarrow \infty$. Indeed, if the nonempty set $\Omega_y(\varphi)$ contains any point besides the origin of R^m , then it contains also a complete trajectory of (3.3) different from the origin because it is semiinvariant with respect to (3.3). But, in consequence of (3.7), this contradicts condition (iii) of the theorem. The proof is complete.

In certain applications condition (ii) in Theorem 3.1 proves to be rather restrictive. For example, it may happen that the potential energy $P(\hat{q}, \tilde{q})$ of a mechanical system is \hat{q} -definite, in every neighbourhood of the origin $\hat{q} = \tilde{q} = 0$ there exists an equilibrium position belonging to the set $P(\hat{q}, \tilde{q}) > 0$, nevertheless the origin is asymptotically \hat{q} -stable (see [1], Examples). Now we relax this condition of the theorem (compare with Theorem 3.3 in [1]).

Theorem 3.2. Suppose that the function Z in (3.2) is bounded on the set G'_y , and there is a Ljapunov function $V: G'_y \rightarrow R$ of (3.2) satisfying conditions (i), (iii) in Theorem 3.1. Assume, in addition, that

(ii') for every $c > 0$, if the set $(\dot{V}_{(3.2)})^{-1}(0) \cap V^{-1}(c)$ contains a complete trajectory of (3.2) then this trajectory is contained in the set $\{(y, z): y=0\}$.

Then the zero solution of (3.2) is asymptotically y -stable.

Proof. We have to modify the proof of Theorem 3.1 only from that point where we assumed $v_0 > 0$. It is enough to prove that in this case $\Omega_y(\varphi) = \{0\}$.

Let $0 \neq q \in \Omega_y(\varphi)$. Then, by Lemmas 3.1—3.2 in [1], either there exists an $r \in R^n$ such that $(q, r) \in \Omega_x(\varphi) \subset M(v_0) = (\dot{V}_{(3.2)})^{-1}(0) \cap V^{-1}(v_0)$ or $|\chi(t_i)| \rightarrow \infty$ whenever $t_i \rightarrow \infty$ and $\psi(t_i) \rightarrow q$ as $i \rightarrow \infty$. In the first case, by the semiinvariance property of $\Omega_x(\varphi)$ with respect to (3.2), the set $M(v_0)$ contains a trajectory of (3.2) not contained in the set $\{(y, z): y=0\}$, which contradicts (ii'). Therefore, if $t_i \rightarrow \infty$ and $\psi(t_i)$ converges to a point different from the origin of R^m , then $|\chi(t_i)| \rightarrow \infty$ as $i \rightarrow \infty$.

We shall prove that in the case $\Omega_y(\varphi) \neq \{0\}$ the inclusion $\Omega_y(\varphi) \subset N(v_0) = V_m^{-1}[v_0, \infty)_0$ holds. But $\Omega_y(\varphi)$ is compact and connected, and $N(v_0)$ is closed, so it is enough to show that $\Omega_y(\varphi) \setminus \{0\} \subset N(v_0)$. Suppose the contrary. Then there exist $q \in \Omega_y(\varphi)$ ($q \neq 0$) and $\varepsilon > 0$ such that $\bar{B}_m(q, 2\varepsilon) \cap [N(v_0) \cup \{0\}] = \emptyset$. We state that

$$(3.8) \quad \alpha = \limsup_{T \rightarrow \infty} \{\dot{V}(\psi(t), \chi(t)): t \geq T, \psi(t) \in \bar{B}_m(q, 2\varepsilon)\} < 0.$$

Indeed, otherwise there is a sequence $\{t_i\}$ for which $t_i \rightarrow \infty$, $\dot{V}(\varphi(t_i)) \rightarrow 0$, $\psi(t_i) \rightarrow q' \in \bar{B}_m(q, 2\varepsilon)$ and, consequently, $|\chi(t_i)| \rightarrow \infty$ as $i \rightarrow \infty$, i.e. $q' \in N(v_0)$, which contradicts the definition of ε . Since V is bounded below, (3.8) implies that $\psi(t) \in \bar{B}_m(q, 2\varepsilon)$ cannot be satisfied on any whole interval $[T, \infty)$. From this fact it follows that there exist sequences $\{t'_i\}$, $\{t''_i\}$ with the properties

$$\begin{aligned} t'_i < t''_i < t'_{i+1}, \quad t'_i \rightarrow \infty; \quad |\psi(t'_i) - q| = \varepsilon, \quad |\psi(t''_i) - q| = 2\varepsilon, \\ \varepsilon \leq |\psi(t) - q| \leq 2\varepsilon \quad (t'_i \leq t \leq t''_i; \quad i = 1, 2, \dots). \end{aligned}$$

Since $Y(\psi(t), \chi(t))$ is bounded, $t''_i - t'_i \geq \beta > 0$ for all i with some constant β and

$$v(t''_i) - v(t'_i) \leq \sum_{j=1}^i \int_{t'_j}^{t''_j} \dot{V}(\varphi(t)) dt \leq i\alpha\beta \rightarrow -\infty,$$

which is a contradiction.

It remains to prove that for every $q \in \Omega_y(\varphi)$ ($q \neq 0$) the system (3.3) has a complete trajectory through q lying in $\Omega_y(\varphi)$. Consider the sequence of the functions $\{\psi^i(t) = \psi(t_i + t)\}$ whose i -th member is a solution of the initial value problem

$$\dot{y} = Y(y, \chi(t_i + t)), \quad y(0) = \psi(t_i) \quad (i = 1, 2, \dots).$$

Since Z is bounded, $|\chi(t_i + t)| \rightarrow \infty$ uniformly with respect to t on each compact interval $[a, b]$ as $i \rightarrow \infty$. Thus, $Y(y, \chi(t_i + t)) \rightarrow Y_*(y)$ uniformly in $(y, t) \in \bar{B}_m(H') \times \times [a, b]$, and $\psi(t_i) \rightarrow q$. Consequently, there exists a subsequence of $\{\psi^i(t)\}$ which

converges uniformly on $[a, b]$ to a solution γ of the initial value problem $\dot{y} = Y_*(y)$, $y(0) = q$ (see [8], p. 297). For each $t \geq 0$ the point $\gamma(t)$ is the limit of a subsequence of $\psi(t + t_i)$. But also $t_i + t \rightarrow \infty$, so $\gamma(t) \in \Omega_y(\varphi)$, which means that $\Omega_y(\varphi)$ contains a complete trajectory of (3.3) different from the origin.

We have proved that if there exists a $q \in \Omega_y(\varphi)$ ($q \neq 0$) then there exists also a complete trajectory of (3.3) different from the origin that is contained by $\Omega_y(\varphi)$ and, because of $\Omega_y(\varphi) \subset N(v_0)$, by $N(v_0)$ as well, in contradiction to assumption (iii). The proof is complete.

Our method can be used for deriving instability theorems, too.

Theorem 3.3. *Suppose that there is a Ljapunov function $V : G'_y \rightarrow \mathbb{R}$ of (3.2) satisfying the following conditions:*

- (i) V is bounded below;
- (ii) for every $\delta > 0$ there exists an $x_0 \in B_k(\delta)$ such that $V(x_0) < 0$;
- (iii) for every $c < 0$ the set $(\dot{V}_{(3.2)})^{-1}(0) \cap V^{-1}(c)$ contains no complete trajectory of (3.2), and
- (iv) the set $V_m^{-1}[c, \infty)_0$ contains no complete trajectory of (3.3).

Then the zero solution of (3.2) is y -unstable.

Proof. We have to prove that there is an $\varepsilon_0 > 0$ such that from every neighbourhood of the origin in \mathbb{R}^k there starts a solution of (3.2) which leaves the set $\bar{B}_m(\varepsilon_0) \times \mathbb{R}^n$.

Let $0 < \varepsilon_0 < H'$. For an arbitrary δ ($0 < \delta < \varepsilon_0$) take an $x_0 \in B_k(\delta)$ such that $V(x_0) < 0$, and consider a solution $x = \varphi(t) = (\psi(t), \chi(t))$ of (3.2) with $\varphi(0) = x_0$. We shall prove that $|\psi(T)| > \varepsilon_0$ for some $T > 0$. Suppose the contrary, i.e. $|\psi(t)| \leq \varepsilon_0$ for all $t \geq 0$. Then $v(t) \rightarrow v_0 < V(x_0) < 0$ as $t \rightarrow \infty$. By Lemma 3.1 in [1] and invariance property of $\Omega_x(\varphi)$, assumption (iii) implies (3.4). As it was shown in the proof of Theorem 3.1, from these facts it follows that the nonempty set $\Omega_y(\varphi)$ is a subset of $V_m^{-1}[v_0, \infty)_0$ (see (3.7)) and it is semiinvariant with respect to (3.3). Consequently, the set $V_m^{-1}[v_0, \infty)_0$ contains at least one complete trajectory of (3.3) in contradiction to assumption (iv) of the theorem. The proof is complete.

Remark 3.1. Let $y = (y_1, y_2)$ be a partition of $y \in \mathbb{R}^m$ ($y_1 \in \mathbb{R}^{m_1}$, $y_2 \in \mathbb{R}^{m_2}$, $1 \leq m_1 < m$, $m_1 + m_2 = m$) and suppose that for some $\varepsilon_0 > 0$ the inequalities $|y|_1 \leq \varepsilon_0$, $V(y_1, y_2, z) < 0$ imply $|y_2| \leq H'$. Analysing the proof of Theorem 3.3 one can easily see that, in fact, in this case the zero solution of (3.2) is y_1 -unstable.

As we shall see in the applications, we often have an estimate of the type $\dot{V}_{(3.2)}(y, z) \leq U(y)$, which allows us to simplify the last condition in Theorems 3.1–3.3. In the following simple proposition even a slightly more general case is considered.

Proposition 3.1. Suppose that for a Ljapunov function $V : G'_y \rightarrow R$ of (3.2) there exists a continuous function $W : G'_y \rightarrow R$ such that

- (i) $\dot{V}_{(3.2)}(y, z) \leq W(y, z) \leq 0 \quad ((y, z) \in G'_y)$;
- (ii) $W(y, z) \rightarrow U(y)$ uniformly in $y \in \bar{B}_m(H')$ as $|z| \rightarrow \infty$.

Then for every $c \in R$,

$$E(c) = U^{-1}(0) \cap V_m^{-1}[c, \infty] \supset V_m^{-1}[c, \infty]_0.$$

4. An application

Consider again the holonomic mechanical system of r degrees of freedom described by the Lagrangian equation

$$(4.1) \quad \frac{d}{dt} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = - \frac{\partial P}{\partial q} + Q \quad (q, \dot{q} \in R^r),$$

where the following notations are used (see [1]): $P(q)$ is the potential energy ($P(0)=0$), $T(q, \dot{q}) = (1/2) \dot{q}^T A(q) \dot{q}$ is the kinetic energy, and $Q(q, \dot{q})$ is the resultant of non-energetic and dissipative forces with complete dissipation.

Let $q = \text{col}(q_1, q_2)$ be a partition of the vector of generalized coordinates ($q_1 \in R^{r_1}$, $q_2 \in R^{r_2}$, $1 \leq r_1 \leq r$, $r_1 + r_2 = r$). Applying our results we give sufficient conditions for asymptotic stability and instability of the equilibrium $q = \dot{q} = 0$ (possibly non-isolated) with respect to the velocities \dot{q} and coordinates q_1 in the case when the system is "asymptotically q_2 -independent". It is worth emphasizing that the coordinates of q_2 are not supposed to be bounded along the motions.

The system (4.1) is defined to be *asymptotically q_2 -independent* if for some constant $H' > 0$ and for every compact set $K \subset R^r$

- (a) there are $\lambda > 0$ and $c \in \mathcal{K}$ such that

$$\lambda |\dot{q}|^2 \leq \frac{1}{2} \dot{q}^T A(q_1, q_2) \dot{q}, \quad Q^T(q_1, q_2, \dot{q}) \dot{q} \leq -c(|\dot{q}|)$$

for all $q_1 \in \bar{B}_{r_1}(H')$, $q_2 \in R^{r_2}$, $\dot{q} \in R^r$;

- (b) $A(q_1, q_2) \rightarrow A_*(q_1)$, $P(q_1, q_2) \rightarrow P_*(q_1)$ as $|q_2| \rightarrow \infty$; in addition, $Q(q_1, q_2, \dot{q}) \rightarrow Q_*(q_1, \dot{q})$ uniformly in $q_1 \in \bar{B}_{r_1}(H')$, $\dot{q} \in K$ as $|q_2| \rightarrow \infty$, as well as $\partial A / \partial q$, $\partial P / \partial q$ converge uniformly in $q_1 \in \bar{B}_{r_1}(H')$ as $|q_2| \rightarrow \infty$.

We are going to apply Theorems 3.2 and 3.3 while $z = q_2$ and V is the total mechanical energy. For this purpose we introduce the Hamiltonian variables $q, p = A(q) \dot{q}$, by the aid of which the system (4.1) can be rewritten in the form

$$(4.2) \quad \begin{aligned} \dot{p} &= -\frac{1}{2} p^T \left(\frac{\partial A^{-1}(q)}{\partial q} \right) p - \frac{\partial P}{\partial q} + Q(q, A^{-1}(q)p), \\ \dot{q} &= A^{-1}(q)p, \end{aligned}$$

In view of asymptotic q_2 -independence, the equilibrium $q = \dot{q} = 0$ of (4.1) and the zero solution $p = q = 0$ of (4.2) have the same stability properties.

Consider the total mechanical energy H defined by $H = H(p, q_1, q_2) = T + P$. As is known (see [8], p. 358),

$$(4.3) \quad \dot{H}_{(4.2)}(p, q_1, q_2) = Q^T(q, A^{-1}(q)p)A^{-1}(q)p \leq -d(|p|)$$

for all $(p, q_1) \in \bar{B}_{r_1+r}(H')$, $q_2 \in R^s$ with a suitable $d \in \mathcal{K}$. Consequently, H is a Ljapunov function of (4.2), and

$$(4.4) \quad (\dot{H}_{(4.2)})^{-1}(0) \cap H^{-1}(c) = \{\text{col}(p, q): P(q) = c, p = 0\} \quad (c \in R),$$

so the trajectories of (4.2) contained in this set are the equilibria $p = 0, q = q_0$ for which $P(q_0) = c$.

Now let us determine the set

$$E(c) = H_{r_1+1}^{-1}[c, \infty] \cap d^{-1}(0) = \{\text{col}(p, q_1): p = 0, q_1 = P_{r_1}^{-1}[c, \infty]\},$$

figuring in Proposition 3.1. Since $\partial P / \partial q_1$ is continuous and converges uniformly as $|q_2| \rightarrow \infty$, the function $P(\cdot, q_2): \bar{B}_{r_1}(H') \rightarrow R$ is continuous uniformly in $q_2 \in R^s$. From this fact it follows that

$$(4.5) \quad E(c) = \{\text{col}(p, q_1): p = 0, P_*(q_1) = c\}.$$

The system (4.2) is asymptotically q_2 -independent, hence its limit system as $|q_2| \rightarrow \infty$ reads as follows:

$$(4.6) \quad \begin{aligned} \dot{p}^i &= -\frac{1}{2} p^T \left(A_{*}^{-1} \frac{\partial A_{*}}{\partial q^i} A_{*}^{-1} \right) p - \frac{\partial P_{*}}{\partial q^i} + Q_{*}^i(q_1, A_{*}^{-1}p) \\ \dot{p}^j &= Q_{*}^j(q_1, A_{*}^{-1}p) \end{aligned}$$

$$\dot{q}^i = \sum_{k=1}^r [A_{*}^{-1}(q_1)]_{ik} p^k$$

for $i = 1, 2, \dots, r_1; j = r_1 + 1, \dots, r$. In view of (4.5), if $E(c)$ contains a trajectory of (4.6) then it is of the form $p = 0, q_1 = (q_1)_0 = \text{const.}$, furthermore

$$(4.7) \quad P_{*}((q_1)_0) = c, \quad \left. \frac{dP_{*}}{dq_1} \right|_{q_1=(q_1)_0} = 0.$$

Theorem 4.1. *Suppose that the mechanical system (4.1) is asymptotically q_2 -independent.*

I. *If (i) the potential energy P is positive q_1 -definite, (ii) system (4.1) has no equilibrium position in the region $\{(q_1, q_2): P(q_1, q_2) > 0, q_1 \neq 0\}$, and (iii) the equality $dP_{*}(q_1)/dq_1 = 0$ implies either $q_1 = 0$ or $P_{*}(q_1) = 0$, then the equilibrium $q = \dot{q} = 0$ of (4.1) is asymptotically (q_1, \dot{q}) -stable.*

II. If (i) the potential energy P has no local minimum at $q=0$, (ii) the system (4.1) has no equilibrium position in the region $\{q: P(q)<0\}$, and (iii) the equality $dP_*(q_1)/dq_1=0$ implies $P_*(q_1)\equiv 0$, then the equilibrium $q=\dot{q}=0$ of (4.1) is q_1 -unstable.

Proof. I. We show that (4.2) and the total mechanical energy H as a Ljapunov function satisfy the conditions of Theorem 3.2. Condition (a) in the definition of the asymptotic q_2 -independence and (i) assure H to be positive (q_1, p) -definite. In consequence of (4.4), for the system (4.2) condition (ii) precludes the possibility of having such a complete trajectory in the set $(\dot{H}_{(4.2)})^{-1}(0) \cap H^{-1}(c)$ ($c>0$) that is not in $\{(q_1, q_2, p): q_1=0, p=0\}$. Finally, using (4.7), condition (ii), and Proposition 3.1 we obtain that the limit system (4.6) cannot have any trajectory in the set $H_{r_1+r}^{-1}[c, \infty)_0$ ($c>0$) except the origin.

II. One can similarly check the conditions of Theorem 3.3, from which (q_1, p) -instability follows. According to Remark 3.1, for the purpose of proving q_1 -instability it is enough to show that $|q_1| \leq \varepsilon_0, H(q_1, p, q_2) < 0$ imply $|p| \leq M$ for some constants $\varepsilon_0 > 0, M$. Observe, that P is bounded below on the set $\bar{B}_{r_1}(\varepsilon_0) \times R^n$ because of q_2 -independence. Therefore, T is bounded above, which together with (a) imply that p belongs to a bounded set. The proof is complete.

Concluding this section we note that in possession of Theorem 4.1 one can easily prove the conjecture made in the Introduction in connection with the motion of a material point on the surface $z=(1/2)y^2[1+1/(1+x^2)]$.

5. A generalization to nonautonomous systems

The LaSalle principle and the invariance property of limit sets with respect to the limiting equation, which served as the two main tools in the proofs of Section 3 have been extended to quite general nonautonomous systems. These extensions enable us to generalize our results to the equation

$$(5.1) \quad \dot{x} = X(x, t) \quad (X(0, t) \equiv 0).$$

Namely, we give a theorem on the partial asymptotic stability of the zero solution of (5.1) without any assumptions on the boundedness of solutions. To formulate and prove it we need some concepts and results from topological dynamics given in [2]—[4]. The theorem will be followed by a corollary, containing only analytical conditions and, consequently, more suitable for applications.

As is known, (5.1) is equivalent to the integral equation $x(t) = x(a) + \int_a^t X(s, x(s)) ds$, i.e. to the functional equation $x = x(a) + I_a x$, where the operator I_a is defined by

$I_a x(t) = \int_a^t X(s, x(s)) ds$. In the method of limiting equation there occur such functional equations in which the operator I_a is more general than the integral with a kernel. An *ordinary integral-like operator* I is a mapping which associates with each continuous function $\varphi: [\alpha, \beta] \rightarrow R^k$ and $a \in [\alpha, \beta]$ a continuous function $I_a \varphi$ so that (1) if $\varphi_i: [\alpha, \beta] \rightarrow R^k$ are continuous and $\varphi_i(t) \rightarrow \varphi(t)$ uniformly, then $I_a \varphi_i(t) \rightarrow I_a \varphi(t)$ uniformly in $t \in [a, b]$, as $i \rightarrow \infty$ for all $[a, b] \subset [\alpha, \beta]$; (2) $I_a \varphi(t) = I_a \varphi(s) + I_s \varphi(t)$ for all $a, s, t \in [\alpha, \beta]$. We shall denote by $u = Iu$ the functional equation $u = u(a) + I_a u$ associated with the ordinary integral-like operator I .

For $t \in R_+$ we define the *translate* X^t of X by $X^t(x, s) = X(x, t+s)$ ($s \in R_+$). We denote by $\text{tran}(X)$ the collection of all translates X^t of X ($t \in R_+$). An ordinary integral-like operator equation $u = Iu$ is a *limiting equation* of (5.1) if there exists a sequence $\{t_i\}$ converging to infinity so that X^{t_i} integrally converges to I as $i \rightarrow \infty$, i.e. whenever $\varphi_i: [a, b] \rightarrow R^k$ converges uniformly to φ then

$$\int_a^b X(\varphi_i(s), t_i + s) ds \rightarrow I_a \varphi(b) \quad (i \rightarrow \infty).$$

The set $\text{tran}(X)$ is said to be *precompact* if every sequence in it has an integrally converging subsequence.

Theorem B. [4] *Suppose that $\text{tran}(X)$ is precompact and $\varphi: [t_0, \infty) \rightarrow R^n$ is a solution of (5.1). Then $\Omega_x(\varphi)$ is semiinvariant with respect to the family of the limiting equations of (5.1), i.e. for each $p \in \Omega_x(\varphi)$ there is a limiting equation $u = Iu$ of (5.1) and a solution γ of the equation $u = p + I_0 u$ so that $\gamma(t) \in \Omega_x(\varphi)$ for all t in the domain of γ .*

By our standard partition $x = (y, z)$ the system (5.1) can be written in the form

$$(5.2) \quad \dot{y} = Y(y, z, t), \quad \dot{z} = Z(y, z, t) \quad ((y, z, t) \in \Gamma_y).$$

Let $0 < H' < H$.

Theorem 5.1. *Suppose that the right-hand sides of (5.2) satisfy the following conditions:*

(i) *for each compact set $K \subset R^n$ and continuous function $\chi: R_+ \rightarrow R^n$ with $|\chi(t)| \rightarrow \infty$ as $t \rightarrow \infty$, there are functions $p, q \in \mathcal{K}$ so that for arbitrary continuous functions $v: [a, b] \rightarrow \bar{B}_m(H')$, $w: [a, b] \rightarrow K$*

$$\left| \int_a^b Y(v(t), \chi(t), t) dt \right| \leq p(b-a), \quad \left| \int_a^b X(v(t), w(t), t) dt \right| \leq q(b-a);$$

(ii) *$\text{tran}(X(x, t))$ is precompact;*

(iii) *$\text{tran}(Y(y, \chi(t), t))$ is precompact for every continuous function $\chi: R_+ \rightarrow R$ with $|\chi(t)| \rightarrow \infty$ as $t \rightarrow \infty$.*

Suppose, in addition, that there is a positive y -definite Ljapunov function $V: \Gamma'_y \rightarrow R$ of (5.2) having the following properties:

(iv) for each $c > 0$ neither limiting equation of (5.2) has a positive semitrajectory in the set $V_k^{-1}[c, \infty)_0$;

(v) for each $c > 0$ and continuous function $\chi: R_+ \rightarrow R^n$ such that $|\chi(t)| \rightarrow \infty$ as $t \rightarrow \infty$, neither limiting equation of $\dot{y} = Y(y, \chi(t), t)$ has a positive semitrajectory in the set $V_m^{-1}[c, \infty)_0$ different from 0.

Then the zero solution of (5.2) is asymptotically y -stable.

Proof. The zero solution of (5.2) is y -stable (see [8], p. 15); therefore, it is sufficient to prove that if $x = \varphi(t) = (\psi(t), \chi(t))$ is a solution of (5.2) and $|\psi(t)| \leq H'' < H'$ for all $t \geq t_0$, then $\psi(t) \rightarrow 0$ as $t \rightarrow \infty$.

Let us introduce the notations $v(t) = V(\varphi(t), t)$ and $v_0 = \lim_{t \rightarrow \infty} v(t)$. We distinguish two cases:

a) Assume that $|\chi(t)| \rightarrow \infty$ as $t \rightarrow \infty$. We show that in this case $v_0 = 0$, which implies $\psi(t) \rightarrow 0$ because V is positive y -definite.

The limit set $\Omega_x(\varphi)$ is not empty and, by Theorem A, $\Omega_x(\varphi) \subset V_k^{-1}[v_0, \infty)_0$. On the other hand, $\Omega_x(\varphi)$ is semiinvariant with respect to the family of the limiting equations of (5.2) (see Theorem B). Consequently, one of them has at least one positive semitrajectory in $V_k^{-1}[v_0, \infty)_0$. Thus, in view of (iv), $v_0 = 0$.

b) Let $|\chi(t)| \rightarrow \infty$ as $t \rightarrow \infty$. We show that either $v_0 = 0$ or $\Omega_y(\varphi) = \{0\}$.

Consider the equation

$$(5.3) \quad \dot{y} = Y(y, \chi(t), t) \quad (y \in \bar{B}_m(H'), t \in R_+)$$

and its Ljapunov function $U(y, t) = V(y, \chi(t), t)$. Using again Theorem A we obtain

$$(5.4) \quad \Omega_y(\varphi) = \Omega_y(\psi) \subset U_m^{-1}[v_0, \infty)_0 \subset V_m^{-1}[v_0, \infty)_0.$$

On the other hand, $\Omega_y(\varphi)$ as the limit set of the solution $y = \psi(t)$ of (5.3) is semiinvariant with respect to the family of the limiting equations of (5.3). If there is a $q \in \Omega_y(\varphi)$, $q \neq 0$, this means that one of the limiting equations of (5.3) has a positive semitrajectory different from $\{0\}$ which is a subset of $\Omega_y(\varphi)$. Then, according to (5.4) and hypothesis (v), $v_0 = 0$. The proof is complete.

Corollary 5.1. Suppose that

(i) for each compact set $K \subset R^n$ there are locally integrable functions $\mu_j, v_j: R_+ \rightarrow R_+$, $j = 1, 2$ so that the functions $\int_0^t \mu_j(s) ds$ are uniformly continuous on R_+ , the functions $\int_t^{t+1} v_j(s) ds$ are bounded on R_+ , and

$$|Y(y, z, t)| \leq \mu_1(t), \quad |Z(w, t)| \leq \mu_2(t),$$

$$|Y(y, z, t) - Y(y', z, t)| \leq v_1(t)|y - y'|, \quad |X(w, t) - X(w', t)| \leq v_2(t)|w - w'|$$

for all $y, y' \in \bar{B}_m(H')$, $z \in R^n$, $w, w' \in \bar{B}_m(H) \times K$, $t \in R_+$. Suppose, furthermore, that there is a positive y -definite Ljapunov function $V: \Gamma'_y \rightarrow R$ of (5.2) having the following properties:

(ii) if for a function $X_*: \Gamma'_y \rightarrow R$ there is a sequence $\{t_i\}$ so that $t_i \rightarrow \infty$ and

$$\int_0^t X(x, s+t_i) ds \rightarrow \int_0^t X_*(x, s) ds \quad (i \rightarrow \infty)$$

for every fixed $(x, t) \in \Gamma'_y$, moreover, if $c > 0$, then the set $V_k^{-1}[c, \infty]_0$ contains no positive semitrajectory of the equation $\dot{x} = X_*(x, t)$;

(iii) if for a function $Y_*: \bar{B}_m(H') \times R_+ \rightarrow R^m$ there exist a sequence $\{t_i\}$ and a continuous function $\chi: R_+ \rightarrow R^n$ so that $t_i \rightarrow \infty$ ($i \rightarrow \infty$), $|\chi(t)| \rightarrow \infty$ ($t \rightarrow \infty$) and

$$\int_0^t Y(y, \chi(s+t_i), s+t_i) ds \rightarrow \int_0^t Y_*(y, s) ds \quad (i \rightarrow \infty)$$

for every fixed $(y, t) \in \bar{B}_m(H') \times R_+$, moreover, if $c > 0$, then the set $V_m^{-1}[c, \infty]_0$ contains no positive semitrajectory of the equation $\dot{y} = Y_*(y, t)$ except the origin $y=0$.

Then the zero solution of (5.2) is asymptotically y -stable.

Proof. As it follows from [2] (Theorem 4.1), under assumption (i) both $\text{tran}(X(x, t))$ and $\text{tran}(Y(y, \chi(t), t))$ are precompact, and all the limiting equations are ordinary differential equations whose right-hand sides are the almost-everywhere derivatives of

$$\lim_{i \rightarrow \infty} \int_0^t X(x, s+t_i) ds, \quad \lim_{i \rightarrow \infty} \int_0^t Y(y, \chi(s+t_i), s+t_i) ds,$$

respectively. This means that all assumptions of Theorem 5.1 are satisfied.

Theorem 5.1 can be used for the case when $X(x, t)$ is periodic in t . For example, if we assume that $Y(y, z, t) \rightarrow Y_*(y, t)$ uniformly in $(y, t) \in \bar{B}_m(H') \times R_+$ as $|z| \rightarrow \infty$, then both $\text{tran}(X(x, t))$ and $\text{tran}(Y(y, \chi(t), t))$ are precompact, and the limiting equations read

$$\dot{x} = X(x, t+t_0), \quad \dot{y} = Y_*(y, t+t_0),$$

respectively.

Remark 5.1. Suppose assumptions (i), (ii), (iv) in Theorem 5.1 to be satisfied. Suppose, in addition, that

(v') for every continuous function $\chi: R_+ \rightarrow R^n$, for which $|\chi(t)| \rightarrow \infty$ as $t \rightarrow \infty$, there is a limiting equation $u = Ju$ of $Y(y, \chi(t), t)$ so that for every $c > 0$ the set $V_m^{-1}[c, \infty]_0$ contains no positive trajectory of $u = u(0) + J_0 u$.

Then the zero solution of (5.2) is equiasymptotically y -stable, i.e. it is y -stable and for every $t_0 \in R_+$ there is a $\sigma(t_0) > 0$ such that $|y(t; x_0, t_0)| \rightarrow 0$ uniformly in $x_0 \in \bar{B}_k(\sigma(t_0))$ as $t \rightarrow \infty$.

To show this we have to modify only part b) of the proof of Theorem 5.1. Namely, we prove that also in this case $v_0 = 0$. After proving (5.4) consider the limiting equation

$$(5.5) \quad u = u(0) + J_0 u.$$

For a sequence $\{t_i\}$ the sequence of translates $Y^{t_i}(y, \chi(t), t)$ tends to J integrally as $i \rightarrow \infty$. From assumption (i) it follows that the functions $\{\psi_i(t) = \psi(t + t_i)\}$ being solutions of the equations $\dot{y} = Y^{t_i}(y, \chi(t), t)$ are uniformly bounded and equicontinuous on every fixed interval $[0, T]$. By Arzela—Ascoli theorem, we can assume that $\psi_i \rightarrow \psi_*$ uniformly on $[0, T]$, thus ψ_* is a solution of (5.5). Obviously, $\psi_*(t) \in \Omega_y(\varphi)$ for all $t \geq 0$. According to (5.4) and assumption (v'), $v_0 = 0$.

So we have proved that $V(x(t; x_0, t_0)) \rightarrow 0$ as $t \rightarrow \infty$ for every fixed $t_0 \in R_+$ and for all x_0 with sufficiently small $|x_0|$. By the classic covering theorem of Heine—Borel—Lebesgue, this convergence is uniform with respect to x_0 [9], which implies equiasymptotic y -stability since V is positive y -definite.

Remark 5.2. The statement in Remark 5.1 remains valid if assumption (v') is weakened so that $V_m^{-1}[c, \infty]_0$ contains no positive semitrajectory of the limiting equation $u = u(0) + J_0 u$ except the origin $y = 0$, but it is supposed, in addition, that $V(y, z, t) \rightarrow 0$ uniformly in $(z, t) \in R^n \times R_+$ as $y \rightarrow 0$.

To see this one has to observe only that the additional condition on V obviously precludes the possibility of $\psi_*(t) \equiv 0$ for the function $\psi_*(t)$ occurring in the argument in Remark 5.1.

These two remarks make it easier to see that our result generalizes and improves the main theorem of [10].

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